# An Algebraic Logic of Partial Functions on Finite Traces

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I will present **a logic defined algebraically**

similar to how

- the algebra of Regular Expressions gives us $\mathsf{LDL}_f$
- the algebra of *-free Regular Expressions gives us $\mathsf{LTL}_f$

But our algebra works with **input data** (relational structures)

It is an **algebra of string-to-string transductions**

(partial functions on strings of relational structures)

$$\textbf{trace} \quad \xrightarrow{\text{algebraic term}} \quad \textbf{new trace}$$

The algebra gives us a **Dynamic Logic**

In the design: **a well-known trade-off**

- want sufficient expressiveness

  e.g., imperative constructs: if-then-else, while

- need to keep the complexity manageable (ideally, P-time)

  since data can be large

**Started from:** FO(LFP), partitioned variables in atomic formulas into I/O, studied how information propagates

Initial work: [T. FROCOS'19], [ABSTV KR'20], [ABSTV ICDT'21]

A **two-level syntax**

$$\alpha ::= m(\varepsilon) \mid \mathrm{id} \overbrace{\mid \curvearrowright\alpha \mid \alpha\,;\alpha \mid \alpha \sqcup \alpha \mid \alpha^{\uparrow} \mid}^{\text{restrictions of } \neg,\wedge,\vee,\ *} (\mathsf{P}{=}\mathsf{Q}) \mid \ \textsc{bg}(\mathsf{P}_{now} \neq \mathsf{Q})$$

all operation are determinism-preserving

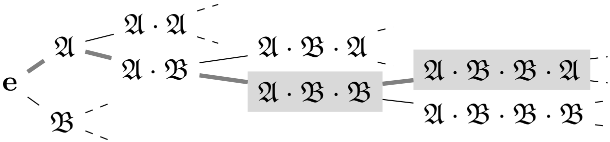
$\varepsilon$ is a **free** function variable ranging over Choice functions

$\tau \ := \ \tau_{\mathrm{EDB}} \uplus \tau_{\mathrm{reg}}$, P, Q $\in \tau_{\mathrm{reg}}$ are "registers"

At the **bottom level**, non-deterministic atomic actions $m \in \mathcal{M}$ are specified by a modification of **unary conjunctive queries**

Let $\mathbf{U}$ denote all $\tau$-structures over the same finite universe

**Semantics** is given via **Choice functions** $h : \mathcal{A} \to (\mathbf{U}^+ \rightharpoonup \mathbf{U}^+)$ assign meaning to atomic actions in $\mathcal{A}$ w.r.t. a transition system **Tr**




$$m^{\textbf{Tr}}(h/\varepsilon)(s_i) = s_{i+1} \;\text{ iff }\; (s_i \mapsto s_{i+1}) \in h(m(\varepsilon))$$

$h$ is extended to all terms: **string-to-string transductions**

## The Algebra is Equivalent to a **Dynamic Logic**

$$\alpha ::= m(\varepsilon) \mid \mathrm{id} \mid \curvearrowright \alpha \mid \alpha \,;\, \alpha \mid \alpha \sqcup \alpha \mid \alpha^{\uparrow} \mid (\mathsf{P}{=}\mathsf{Q}) \mid \textsc{bg}(\mathsf{P}_{now} \neq \mathsf{Q}) \mid \varphi?$$
$$\varphi ::= \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid |\alpha\rangle\varphi$$

The state formulae in the second line are shorthands:

$$\top := \mathrm{id}$$
$$\neg\varphi := \curvearrowright \varphi$$
$$\varphi \wedge \psi := \varphi \,;\, \psi$$
$$\varphi? := \curvearrowright\curvearrowright \varphi \qquad = \mathit{Dom}(\varphi) \qquad \text{(test action)}$$
$$|\alpha\rangle\, \varphi := \mathit{Dom}(\alpha \,;\, \varphi)$$

**Semantics** of process expressions is as before

For state formulae we have:

$$\mathbf{Tr}, s \models \varphi(h/\varepsilon) \quad \text{iff} \quad \varphi^{\mathbf{Tr}}(h/\varepsilon)(s) = s.$$

In particular, the string $s$ may contain just one structure $\mathfrak{A}$

**Main Query** with free variable $\varepsilon$

$$\mathbf{Tr}, \mathfrak{A} \models |\alpha\rangle\top(\varepsilon)$$

returns a set of Choice functions

(**Tr** is fixed, so can be omitted)

A **computational problem specified by $\alpha$**

is an isomorphism-closed class $\mathcal{P}_\alpha$ of $\tau$-structures $\mathfrak{A}$ such that there exists $h$ such that

$$\mathfrak{A} \models |\alpha\rangle\top \,(h/\varepsilon)$$

**Certificates**: equivalence classes of Choice functions

Problem: **Size Four** $\alpha_4$
Given: A structure $\mathfrak{A}$ with an empty vocabulary.
Question: Is $|dom(\mathfrak{A})|$ equal to 4?

$$\alpha_4 := \mathit{GuessNewP}^4;\ \curvearrowright \mathit{GuessNewP},$$

$$\mathit{GuessP}(\varepsilon) := \{\ P(x) \longleftarrow\quad \}$$

$$\mathit{GuessNewP}(\varepsilon) := \mathit{GuessP}\,;\ \textsc{bg}(P_{now} \neq P).$$

The answer to the query $\mathfrak{A} \models |\alpha_4\rangle \top(\varepsilon)$, is non-empty, iff the input domain is of size 4

**Other Cardinality examples**: Same Size, EVEN, ...

**Programming constructs** are **definable**

$$
\begin{aligned}
&\textbf{if } \varphi \textbf{ then } \alpha \textbf{ else } \beta \ := \ (\varphi? \,;\, \alpha) \sqcup \beta \\
&\textbf{while } \varphi \textbf{ do } \alpha := (\varphi? \,;\, \alpha)^{\uparrow} \,;\, (\curvearrowright \varphi?) \\
&\textbf{repeat } \alpha \textbf{ until } \varphi := \alpha \,;\, ((\curvearrowright \varphi?) \,;\, \alpha)^{\uparrow} \,;\, \varphi?
\end{aligned}
$$

the full power of regular expressions ($\cup$, $*$, $^c$) is not needed

Problem: **s-t-Connectivity** $\alpha(E, S, T)$
Given: Binary relation $E$, two constants $s$ and $t$, as singleton-set relations $S$ and $T$.
Question: Is $t$ reachable from $s$ by following the edges?

$$\alpha(E, S, T) := \quad M_{base\_case} \,;\, \textbf{repeat} \, \big(M_{ind\_case}; \\ {}_{\textsf{BG}}(\textit{Reach}' \neq \textit{Reach})\big) \,;\, \textit{Copy} \;\; \textbf{until} \; \textit{Reach} = T.$$

$$\begin{aligned} M_{base\_case}(\varepsilon) \quad &:= \quad \{ \; \textit{Reach}(x) \leftarrow S(x) \; \}, \\ M_{ind\_case}(\varepsilon) &:= \{ \; \textit{Reach}'(y) \leftarrow \textit{Reach}(x), E(x, y) \; \}, \\ \textit{Copy}(\varepsilon) &:= \quad \{ \; \textit{Reach}(x) \leftarrow \textit{Reach}'(x) \; \}. \end{aligned}$$

The answer to the query $\mathfrak{A} \models |\alpha\rangle\top(\varepsilon)$, is non-empty, iff s-t reachability holds

**Other examples**: Same Generation, mod 2 Linear Equations . . .

Analyze **data complexity** [Vardi:1982] of

$$\mathfrak{A} \models |\alpha\rangle\top\ (\varepsilon)$$

**Theorem:** The logic restricted to simple tests captures NP

For the full logic, data complexity is in PSPACE

To **summarize**, we presented an algebra (Dynamic Logic) interpreted as partial functions on strings of relational structures

linear-time, finite traces, with complex nested tests

Current Work: algebraic conditions for P-time data complexity

proof system (cyclic proofs),

(complexity of) validity, satisfiability and containment problems

Future Work: exact expressiveness of the propositional fragment

expression complexity, strategic reasoning,

normative reasoning, preferences, BPM, ...

Thank you!

## **Semantics**, informally

Negation (Anti-Domain): $\curvearrowright \alpha$ – there is no outgoing $\alpha$-transition

Composition: $\alpha \, ; \beta$ – execute sequentially

Preferential Union: $\alpha \sqcup \beta$ – perform $\alpha$ if it's defined, o.w. perform $\beta$

Maximum Iterate: $\alpha^{\uparrow}$ – output the longest transition of $\alpha^*$

Comparison for Equality: P=Q – compare the "content of" P,Q

Back Globally $\textsc{bg}(\mathrm{P}_{now} \neq \mathrm{Q})$ – the "content of" P is new compared to Q earlier in the computation